Confinement and chiral symmetry breaking via a domainlike mean field.

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Singular gauge fields in the partition function for QCD can lead to a domain-like picture for the QCD vacuum by virtue of constraints on quantum fluctuations at the singularities. With a simple model of hyperspherical domains with interiors of constant field strength we show that the main features of gluon condensation and an area law for static quarks can be realised. The Dirac operator in such a background is exactly soluble. Chirality properties of the solutions show agreement with recent lattice results.

1. INTRODUCTION

The nature and consequences for confinement and chiral-symmetry breaking of long-range gluon fields in the vacuum have been the subject of extended investigation. While many studies rely on one or more of the specific configurations of monopoles, vortices and instantons, there are arguments [1] that in fact the complete hierarchy of singular gauge fields must play a role, especially if these objects condense in the vacuum. It is natural then to seek a model in which the bulk properties of this hierarchy can be effectively represented. We partially do this here by introducing a model for such fields in which singularities in vector potentials are concentrated on three-dimensional hypersurfaces ∂V_j in Euclidean space, in the vicinity of which gauge fields can be divided into a sum of a singular pure gauge S_μ and a regular fluctuation part Q_μ, and where a certain $SU(3)_{\rm colour}$ colour vector n_j^a can be associated with S_μ. Thus we have idealised the situation by ignoring lower dimensional singular objects which would appear as "dislocations" at the abelian domain wall singularities.

The requirement of finiteness of the action (density) for singular background plus fluctuation parts for configurations dominating the functional integral [2] can be formalised in our case as specific boundary conditions on ∂V_j for the fluctuation fields charged with respect to n_j. The interiors of these regions thus constitute "domains" V_j. Gauge field modes neutral with respect to n_j^a are not restricted and generate interactions between domains.

Within the present framework then, we decompose a general gauge field $A_\mu^j = S_\mu^j + Q_\mu^j$ and demanding finiteness of the classical action density at the singularity we come to the conditions

$$\breve{n}_j Q_\mu^{(j)} = 0, \ x \in \partial V_j,$$

for gluons, and

$$\psi = -i\, \not\eta^j e^{i\alpha_j\gamma_5}\psi, \bar{\psi} = \bar{\psi} i\, \not\eta^j e^{-i\alpha_j\gamma_5}, x \in \partial V_j \quad (1)$$

for quarks. The adjoint matrix $\breve{n}_j = T^a n_j^a$ appears in the condition for gluons, and a bag-like boundary condition arises for quarks, with a unit vector $\eta_\mu^j(x)$ normal to ∂V_j.

In a given domain V_j the effect of fluctuations in neighbouring regions can be seen as an external gauge field $B_{j\mu}^a$ neutral with respect to n_j^a. This enables an approximation in which different domains are assumed to be decoupled from each other but, to compensate, a certain mean field is introduced in domain interiors. To make the model analytically tractable we consider spherical

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domains with fixed radius R and approximate the mean field in V_j by a configuration with the field strength $\hat{\mathcal{B}}^{(j)a}_{\mu\nu} = \hat{n}^{(j)} B^{(j)}_{\mu\nu}$, which is (anti-)self-dual $\tilde{B}^{(j)}_{\mu\nu} = \pm B^{(j)}_{\mu\nu}$ so that $B^{(j)}_{\mu\nu} B^{(j)}_{\rho\nu} = B^2\delta_{\mu\rho}$, with B a constant the same for all domains. The constant colour matrix $\hat{n}^{(j)} = t^3\cos\xi_j + t^8\sin\xi_j$ belongs to the Cartan subalgebra with angles $\xi_j \in \{\frac{\pi}{6}(2k+1),\ k = 0,\dots,5\}$ corresponding to the Weyl subgroup. There is no source for this field on the boundary and therefore it should be treated as strictly homogeneous in all further calculations. The homogeneity itself appears here just as an approximation. Thus the partition function we will deal with can be written

$$\begin{aligned}\mathcal{Z} &= \mathcal{N} \lim_{V,N\to\infty} \prod_{i=1}^{N} \int_V \frac{d^4 z_i}{V} \int\limits_{\Sigma} d\sigma_i \int \mathcal{D}Q^i \\ &\quad \times \mathcal{D}\psi_i \mathcal{D}\bar{\psi}_i \delta[D(\breve{\mathcal{B}}^{(i)})Q^{(i)}]\Delta_{\rm FP}[\breve{\mathcal{B}}^{(i)}, Q^{(i)}] \\ &\quad \times \exp\left\{-S^{\rm QCD}_{V_i}\left[Q^{(i)} + \mathcal{B}^{(i)}, \psi^{(i)}, \bar{\psi}^{(i)}\right]\right\}\end{aligned}$$

where the gluons and quarks are integrated over the spaces $\mathcal{F}^i_Q$ and $\mathcal{F}^i_\psi$ respectively, and the thermodynamic limit assumes $V, N \to \infty$ but with the density $v^{-1} = N/V$ taken fixed and finite. The measure of integration over parameters characterising domains is defined as

$$\begin{aligned}\int\limits_{\Sigma} d\sigma_i \dots &= \frac{1}{48\pi^2} \int\limits_0^{2\pi} d\alpha_i \int\limits_0^{2\pi} d\varphi_i \int_0^{\pi} d\theta_i \sin\theta_i \\ &\quad \times \int_0^{2\pi} d\xi_i \sum_{l=0,1,2}^{3,4,5} \delta(\xi_i - \frac{(2l+1)\pi}{6}) \\ &\quad \times \int_0^{\pi} d\omega_i \sum_{k=0,1} \delta(\omega_i - \pi k) \dots . \quad (2)\end{aligned}$$

Here φ_i and θ_i are spherical angles of the chromomagnetic field, ω_i is the angle between the chromomagnetic and chromoelectric fields, ξ_i is the angle in the colour matrix $\hat{n}_i$, α_i is the chiral angle and z_i is the centre of the domain V_i.

This partition function describes a statistical system of density N/V composed of noninteracting extended domain-like structures, each of which is characterised by a set of internal parameters and whose internal dynamics are represented by the fluctuation fields.

2. MEAN FIELD CORRELATORS

In this model the connected $n-$point correlator of field strength tensors,

$$B^a_{\mu\nu}(x) = \sum_j^N n^{(j)a} B^{(j)}_{\mu\nu} \theta(1 - (x - z_j)^2/R^2),$$

can be calculated explicitly using the measure, Eq. (2). Translation-invariant functions

$$\Xi_n(x_1, \dots, x_n) = \frac{1}{v} \int d^4 z \prod_{i=1}^{n} \theta(1 - \frac{(x_i - z)^2}{R^2})$$

emerge and can be seen as the volume of the region of overlap of n hyperspheres of radius R and centres $(x_1, \dots, x_n)$, normalised to the volume of a single hypersphere $v = \pi^2 R^4/2$. The functions Ξ_n are continuous and vanish if $|x_i - x_j| \geq 2R$. Correlations in the background field have finite range $2R$. The Fourier transform of Ξ_n is then an entire analytical function and thus correlations do not have a particle interpretation. The statistical ensemble of background fields is not Gaussian since all connected correlators are independent of each other and cannot be reduced to the two-point correlations.

The simplest application of the above correlators gives a gluon condensate density which to this approximation is $g^2\langle F^a_{\mu\nu}(x) F^a_{\mu\nu}(x)\rangle = 4B^2$.

Another vacuum parameter which plays a significant role in the resolution of the $U_A(1)$ problem is the topological susceptibility [3]. To define this we consider first the topological charge density $Q(x) = \frac{g^2}{32\pi^2}\tilde{F}^a_{\mu\nu}(x) F^a_{\mu\nu}(x)$ which to lowest order is

$$Q(x) = \frac{B^2}{8\pi^2} \sum_{j=1}^{N} \theta[1 - (x - z_j)^2/R^2] \cos\omega_j,$$

where $\omega_j \in \{0, \pi\}$ depends on the duality of the j-th domain. For a given field configuration the topological charge is additive $Q = \int_V d^4 x Q(x) = q(N_+ - N_-)$ for $-Nq \leq Q \leq Nq$ where $q = \frac{B^2 R^4}{16}$ is a 'unit' topological charge, namely the absolute value of the topological charge of a single domain, and N_+ (N_-) is the number of domains with (anti-)self-dual field, $N = N_+ + N_-$. The probability of finding the topological charge Q in a given configuration is defined by the ratio of the number of configurations $\mathcal{N}_N(Q)$ with a charge Q and the total number of configurations $\mathcal{N}_N$,

$$\mathcal{P}_N(Q) = \frac{N!}{2^N\,(N/2-Q/2q)!\,(N/2+Q/2q)!}.$$

The distribution is symmetric about $Q = 0$, where it has a maximum for N even. For N odd the maximum is at $Q = \pm q$. Averaged topological charge is zero.

The topological susceptibility χ is determined by the two-point correlator of topological charge density, which in the lowest approximation gives

$$\chi = \int d^4x \langle Q(x)Q(0)\rangle = \frac{B^4R^4}{128\pi^2}. \tag{3}$$

3. AREA LAW

To zeroth order in fluctuations the Wilson loop can be written

$$\begin{aligned} W(L) \quad &= \lim_{V,N\to\infty}\prod_{j=1}^{N}\int_V \frac{d^4z_j}{V}\int d\sigma_j \\ &\times\frac{1}{N_c}\mathrm{Tr}\exp\left\{i\int_{S_L} d\sigma_{\mu\nu}(x)\hat{B}_{\mu\nu}(x)\right\} \end{aligned}$$

where Stokes' theorem has been used. Note that path ordering in our case is not necessary since the matrices $\hat{n}^k$ are assumed to be in the Cartan subalgebra. Computationally it is convenient to consider a circular contour in the (x_3, x_4) plane of radius L with centre at the origin. Details of the calculation are given in [4]. Put briefly, the colour trace is evaluated exactly followed by the integration over orientations of the vacuum field. Integrating over the domain centres and taking the thermodynamic limit ($N\to\infty$, $v = V/N = \pi^2R^4/2$), one obtains for a large Wilson loop $L \gg R$

$$W(L) = \lim_{N\to\infty}\left[1-\frac{1}{N}U(L)\right]^N = e^{-U(L)}$$

with the exponent $U(L) = \sigma\pi L^2 + O(L)$ displaying an area law with string tension for $SU(3)_{\rm colour}$ given by $\sigma = Bf(BR^2)$ with the function

$$f(z) = \frac{2}{3\pi z}\left(3 - \frac{\sqrt{3}}{2\pi z}\int_0^{2\pi z/\sqrt{3}}\frac{dx}{x}\sin x - \frac{2\sqrt{3}}{\pi z}\int_0^{\pi z/\sqrt{3}}\frac{dx}{x}\sin x\right)$$

having a purely geometrical origin. This function is positive for $z > 0$ and has a maximum for $z = 1.55$. We choose this maximum to estimate the model parameters by fitting the string constant to the lattice result,

$$\sqrt{B} = 947\mathrm{MeV},\ \ R^{-1} = 760\mathrm{MeV}, \tag{4}$$

and get for the gluonic parameters of the vacuum $\sqrt{\sigma} = 420$MeV, $\chi = (197\mathrm{MeV})^4$, $\frac{\alpha_s}{\pi}\langle F^2\rangle = 0.081(\mathrm{GeV})^4$ and $q = 0.15$, while the density of the system is $42\mathrm{fm}^{-4}$. There is no separation of characteristic scales $\sqrt{B}R \approx 1$, hence an approximation based on large or small domains is not justifiable and has been not used here.

If B goes to zero then the string constant vanishes. This underscores the role of the gluon condensate in the confinement of static charges. On the other hand, if the number of domains is fixed and the thermodynamic limit is defined as $V, R\to\infty, N =$ const. $<\infty$, namely if the domains are macroscopically large, then $W(L) = 1$, which indicates the absence of a linear potential between infinitely massive charges in a purely homogeneous field.

An area law obviously does not occur for adjoint charges because of the presence of zero eigenvalues of the adjoint matrix n^aT^a.

4. DYNAMICAL CONFINEMENT

Confinement of the fluctuation fields can be seen in the analytical properties of their propagators. For the above boundary conditions the fluctuation quark and gluon propagators can be analytically calculated by reduction to the charged scalar field problem. This in turn is essentially just that of a four-dimensional harmonic oscillator with the orbital momentum coupled to the external field, and the general solution has been found exactly by decomposition over hyperspherical harmonics [4].

Due to the above boundary conditions the $x-$space propagators of charged fields are defined in regions of finite support where they have integrable singularities so that their Fourier transforms are entire functions in the complex momentum plane. This is consistent with a confinement of dynamical fields[5]. Entire propagators have the physically appealing property of a Regge spectrum of relativistic bound states[6].

5. CHIRALITY OF QUARK MODES

Turning to quarks specifically, we address the eigenvalue problem for the massless Dirac operator in a domain,

$$(i\not{D}-\lambda)\psi(x)=0,$$
$$D_\mu=\partial_\mu-i\hat{B}_\mu=\partial_\mu+\frac{i}{2}\hat{n}B_{\mu\nu}x_\nu,$$

subject to the boundary condition Eq.(1). We introduce projectors in the colour direction $N_\pm=\frac{1}{2}(1\pm\hat{n}/|\hat{n}|)$ and spin projection with respect to the magnetic field $\Sigma_\pm=\frac{1}{2}(1\pm\vec{\Sigma}\vec{B}/B)$ with $\hat{B}=|\hat{n}|B$. It is also useful to introduce the mixed projector $O_\zeta=N_+\Sigma_\zeta+N_-\Sigma_{-\zeta}$. Substituting

$$\begin{aligned}\psi &= (i\not{D}+\lambda)\Phi(x),\\ \Phi &= P_\pm\Phi_0+P_\mp O_+\Phi_{+1}+P_\mp O_-\Phi_{-1}\end{aligned}\qquad(5)$$

shows that Φ_ζ satisfies

$$(-D^2+2\zeta\hat{B}+\lambda^2)\Phi_\zeta=0,$$

whose exact solution is given in [4]. It turns out that it is impossible to fulfill the boundary condition Eq.(1) with the term Φ_0 in Eq.(5). Thus Φ must have definite chirality corresponding to the duality of the vacuum gluon field.

The boundary condition enforces a discretisation of the eigenvalues λ. The states are then labelled by the set of quantum numbers: $\mathcal{N}=n,k,m_1,m_2,\zeta,s$ with $n=0,1,\ldots$ a radial quantum number labelling the solutions of the boundary condition for given $O(4)=O(3)\times O(3)$ angular quantum numbers k,m_1,m_2, $\zeta=\pm1$ corresponding to the colour projection with respect to $\hat{n}$ and $s=\uparrow,\downarrow$ representing the spin projection with respect to the magnetic field. Zero modes are absent under chosen boundary conditions. The eigenspinors have the structure

$$\psi_{\mathcal{N}}=i\not{\eta}\chi_{\mathcal{N}}+\varphi_{\mathcal{N}},\gamma_5\chi_{\mathcal{N}}=\mp\chi_{\mathcal{N}},\gamma_5\varphi_{\mathcal{N}}=\mp\varphi_{\mathcal{N}},$$

where spinors $\varphi_{\mathcal{N}}$ and $\chi_{\mathcal{N}}$ as well as the constraints defining the corresponding eigenvalues are known in analytical form and will be discussed in detail elsewhere [7]. For illustration we give the eigenvalue equations for the purely radial ($k=m_1=m_2=0$) modes. With $\Lambda=\lambda/\sqrt{2\hat{B}}$, $z=\hat{B}r^2/2$ we have the equation for $\Lambda^{\downarrow+}_{n000}$

$$\sqrt{z}\Lambda M\left(\Lambda^2+1,3,z\right)+2e^{\mp i\alpha}M\left(\Lambda^2,2,z\right)=0$$

and that for $\Lambda^{\downarrow-}_{n000}$

$$\sqrt{z}\left[\frac{\Lambda^2+2}{2}M\left(\Lambda^2+3,3,z\right)-M\left(\Lambda^2+2,2,z\right)\right]$$
$$+e^{\mp i\alpha}\Lambda M\left(\Lambda^2+2,2,z\right)=0$$

and $\Lambda^{\uparrow+}_{n000}=\Lambda^{\downarrow-}_{n000}$ and $\Lambda^{\uparrow-}_{n000}=\Lambda^{\downarrow+}_{n000}$. Here $M(a,b,z)$ is the confluent hypergeometric function. The sign minus (plus) in front of $i\alpha$ corresponds to the (anti-)self-dual domain.

With these solutions we investigate the chirality of the eigenmodes and also how their probability density correlates with the underlying domain. Following [8] we introduce the local chirality parameter $X(x)$ defined via

$$\tan\left(\frac{\pi}{4}(1+X(x))\right)=\frac{|\psi_L(x)|}{|\psi_R(x)|},$$

which will give extremal values $X=\pm1$ at positions x where $\psi(x)$ is purely right(left) handed. For lattice overlap fermions, even in uncooled gauge fields, histograms of X measured at probability density maxima for low-lying overlap-Dirac eigenmodes show peaks close to the extremal values [9]. This indicates that low-lying modes are strongly chiral and are thus a useful filter for the duality of "objects" underlying the gauge field fluctuations. Within the present model we observe that at the centres of domains all of the eigenmodes are exactly chiral and probability densities are maximal. The "width" of the peaks for the lowest modes at half-maximum varies for different values α and is of the order of $.12-0.14$fm, consistent with the lattice observations of [10]. As is seen from Fig.(1) the chirality of the lowest mode ($n=0$) monotonically decreases with distance from the centre. The chirality parameter for the excited modes alternates between extremal values, the number of alternations is correlated with the radial number n, and

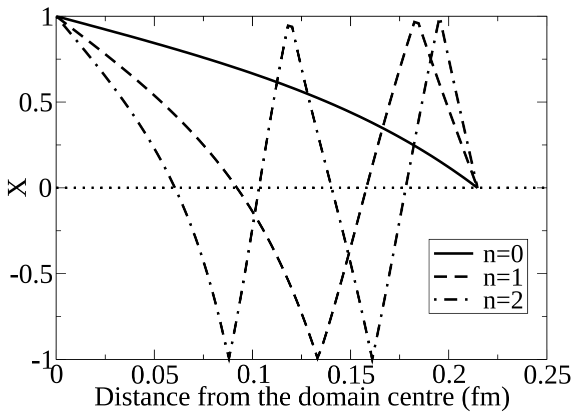

Figure 1. Chirality parameter for the lowest radial modes $\psi_{n000}^{\downarrow+(\uparrow-)}$, self-dual domain, $\alpha=\pi/2$.

the half-width decreases with growing n. The chirality parameter is zero at the boundary for all modes. Qualitatively this picture does not depend on angle α.

The degree of chirality of the modes can be conveniently characterized via a histogram if we average X(x) over a small neighborhood of the domain centre. Then, by assuming that all values of α are equally probable we compute the probability to find a given value of smeared X among a set of modes. The result given in Fig.(2) was obtained for the lowest modes with all possible spin-color orientaions. We observe the double peaking close to the extrema with $X \approx \pm 0.8$. Including higher modes will broaden the peaks and build up a central plateau. This feature as well as the above mentioned values for the half-decay and the density of domains is in qualitative and quantitative agreement with recent lattice results [9,10]. However, at this stage, it would be premature to go beyond purely numerical comparison of our results and those of the lattice. Finally, we quote the estimate of the quark condensate density averaged over (anti-)self-dual configurations and α obtained in [4] directly from the quark propagator. For B and R as in Eq. (4) the density at the domain center is equal to $\langle\bar{\psi}\psi\rangle_{\rm center} = -(228{\rm MeV})^3$. It should be stressed that this nonzero result is due to the non-integral topological charge q associated with domains. This and our results for the chirality

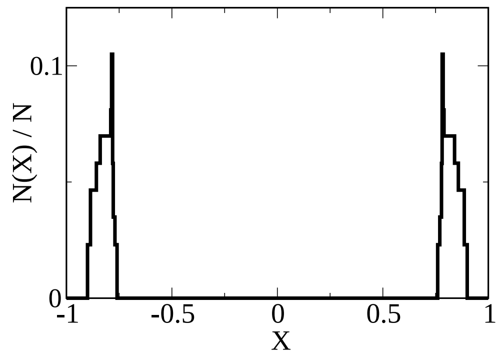

Figure 2. Histogram of chirality parameter X averaged over the central region with radius 0.05fm.

parameter X suggest that the underlying mechanism of chiral symmetry breaking here is the strong chirality of the low-lying non-zero modes, which in turn is a consequence of the (anti-)self-duality of the background gluonic configurations and their non-integral topological charge.

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